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Exact Solutions and Analysis for a Class of Extended Stokes' Problems

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Abstract

This paper studies a class of unsteady flows as extensions of the classic Stokes' problems to consider influence of solid walls, effect of pressure gradients, and situation of two-layer fluids. The flows are solved using the method of separating variables and the eigenfunction expansion method. With simplifications, the derived solutions will degenerate to solutions to the classic Stokes' problems, the Couette flow, and the Poiseuille flow. The exact solutions of these flows clearly illustrate the complexity of the involved physics including evolution of flow velocity profiles and energy transferring at fluid boundaries. For a single-layer flow driven by a plate moving at a constant speed, energy transferred from the plate decreases with time and tends to a non-zero constant as a result of wall effect. In a single-layer flow with an oscillatory boundary, negative energy input may appear at the boundary. For an air-water flow with a finite depth, the interface velocity is proportional to the air velocity, which is a well-known observation in physical oceanography. In addition, there is no energy transferring at the interface between the two fluids in a purely pressure driven two-layer flow.

1 Introduction

The Stokes' problems are classic problems of flows driven by tangential velocity at fluid boundaries [1–3]. The Stokes' first problem is a transient flow set by sudden movement of an infinite plate at a constant speed, while the Stokes' second problem is a flow generated by a periodic motion of a plate [2, 3]. The Stokes'

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problems are unsteady, purely shear flows, and they involve fundamental phenomena of fluid flows. It is significant to investigate such problems and their extensions since they shed insight in understanding of many actual flows, such as water currents in rivers, lakes, and oceans [4], water flows in settling tanks of wastewater treatment plants [5], and liquid flows in micro-electronic devices [6].

Both the classic Stokes' first and the second problems have been analytically solved, and the solutions are among a few available exact solutions to the Navier-Stokes equations [2, 3, 7]. Due to their significance in various backgrounds, a number of variants and generalizations of the problems have also been analyzed. Among others, examples are exact solutions for modified Stokes' problems with influence of solid walls and effects of wall slipeage [8]. An analytical solution is derived for an impulsively started flow driven by a prescribed shear stress rather than a given velocity as in the classic Stokes' problem [9]. Relations among solutions for the Stokes problems, the Couette flow, the Poiseuille flow, and boundary layers are discussed [10]. Recently, increasing efforts are made on extended Stokes' problems with non-Newtonian and magnetic effects [11, 12]. Nevertheless, more research is needed on extended Stokes' problems, especially those with wall effects, pressure gradients, and presence of two-layer fluids such as air-water flows at coastal water surfaces [4].

In this paper, we study a class of problems as extensions of the classic Stokes' problems. Not only the extensions include influence of solid walls and effects of pressure gradient, but also they consider two-layer flows, which, to the best knowledge of the authors, have not be previously studied. Exact solutions of the extended problems are derived using the method of separating variables and the eigenfunction expansion method, and they are analyzed to provide insight for involving physics, in particular, evolution of velocity profiles and power transferring at boundaries of fluids. The derived solutions will reduce to the solutions to the classic Stokes' problems, the Couette flow, and the Poiseuille flow under corresponding simplifications.

The rest of this paper is organized as follows. Section 2 studies single-layer flows with a finite depth. Evolution of a two-layer flow with an initial velocity is analyzed in Section 3. Section 4 investigates influence of pressure gradient. A two-layer flow generated by the motion of a solid wall is studied in Section 5. Section 6 concludes the paper.

2 Single-layer flows

Consider a flow between two horizontal plates (Fig. 1). The fluid is initially static and then starts to move due to the motion of the upper plate at a prescribed speed. This flow is described as the following initial-boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nu \dfrac{\partial^2 u}{\partial y^2}, \\ u(y,0) = 0, \\ u(0,t) = V(t), \quad u(-h,t) = 0. \end{cases} \tag{1}$$

Here t is the time, y is the coordinate in the cross-section direction of the flow, u is the flow velocity, $V(t)$ is the velocity of the upper plate, h is the width between the two plates, and ν is the kinematic viscosity.

In order to solve problem (1), we introduce

$$v(y,t) = u(y,t) - V(t)(1 + \frac{y}{h}). \tag{2}$$

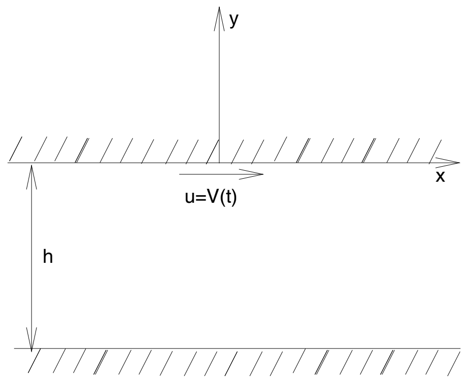

Fig. 1 A schematic representation of a single-layer flow.

It can be shown that $v(y,t)$ satisfies the following governing equation and homogeneous boundary conditions

$$\begin{cases} \frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial y^2} - V'(t)(1+\frac{y}{h}), \\ v(y,0) = -V(0)(1+\frac{y}{h}), \\ v(0,t) = v(-h,0) = 0. \end{cases} \tag{3}$$

First, considering the homogeneous form of this problem, or, ignoring the source with $V'(t)$ in its governing equation, and using the method of separating variables, it can be shown that the eigenfunctions for problem (3) is $\{\sin(n\pi y/h)\}$ [13]. Second, within the space spanned by the eigenfunctions, expressing $v(y,t)$ as

$$v(y,t) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi y}{h} \tag{4}$$

and making the following expansion with the aid of the eigenfunctions [13]

$$1 + \frac{y}{h} = -\sum_{n=1}^{\infty} \frac{2}{n\pi} \sin \frac{n\pi y}{h}, \tag{5}$$

from problem (3) we derive an initial value problem of an ordinary differential equation as follows

$$\begin{cases} T_n'(t) = -\nu(\frac{n\pi}{h})^2 T_n(t) + \frac{2V'(t)}{n\pi}, \\ T_n(0) = \frac{2V(0)}{n\pi}. \end{cases} \tag{6}$$

This problem is readily solved as

$$T_n(t) = \frac{2}{n\pi}[V(t) - \nu(\frac{n\pi}{h})^2 \int_0^t V(\tau) e^{-\nu(\frac{n\pi}{h})^2(t-\tau)} d\tau]. \tag{7}$$

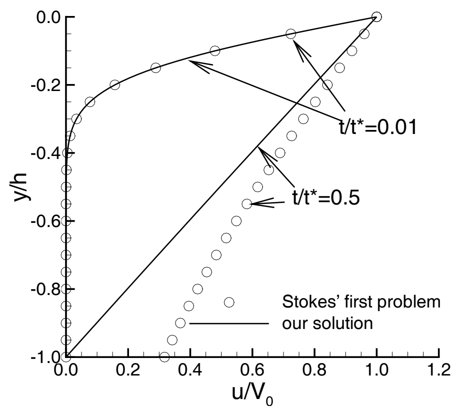

Fig. 2 Water flow driven by a wall moving at a constant speed. $t^* = h^2/\nu$, $\nu = 1.12\times10^{-6}$ m^2/s, $h = 1$ m , and $V_0 = 1$ m/s.

From Eqs. (2), (4) and (7), finally we can find the solution to problem (1) as

$$u(y,t) = V(t)\left(1+\frac{y}{h}\right) + \sum_{n=1}^{\infty} T_n(t)\sin\frac{n\pi y}{h}. \tag{8}$$

Remark 1. Problem (1) is an extension of the classic Stokes' problems since it may reduce to the Stokes' first and second problem under corresponding simplifications. In contrast to the classic Stokes' problems, problem (1) considers a more general form of $V(t)$ and wall effects. Moreover, the Stokes' second problem considers a periodic flow, while problem (1) includes a transient stage before its periodic pattern also. When $h \to \infty$ and with corresponding expressions for $V(t)$, solution (8) will reduce to solutions of these two problems. In addition, solution (8) will become a solution for the Couette flow as $t \to \infty$.

Let us consider two simplifications of solution (8). The first one occurs when $V(t) = V_0$, or, the top wall velocity is a constant, and the solution becomes

$$u(y,t) = V_0\left(1+\frac{y}{h}\right) + \sum_{n=1}^{\infty} \frac{2V_0}{n\pi} e^{-\nu(\frac{n\pi}{h})^2 t}\sin\frac{n\pi y}{h}. \tag{9}$$

The temporal evolution of solution (9) is shown in Fig. 2. For comparison, the solution to the classic Stokes' first problem is also plotted in the figure, which reads as [2]

$$u(y,t) = V_0[1-\mathrm{erf}(\frac{-y}{2\sqrt{\nu t}})]. \tag{10}$$

Notice here $y < 0$, as shown in Fig. 2. When time is small, the effect of the bottom wall is basically not in presence, and the two solutions agree well with each other. However, as time becomes big enough, the bottom wall plays a role; it is seen that solution (9) remains to be zero at the bottom of the plate and it deviates from that of Stokes' problem. From the figure it is also seen that the distribution of the velocity tends to a straight line as $t \to \infty$. Actually, it is anticipated that the flow becomes the classic steady Couette flow as time goes to infinity.

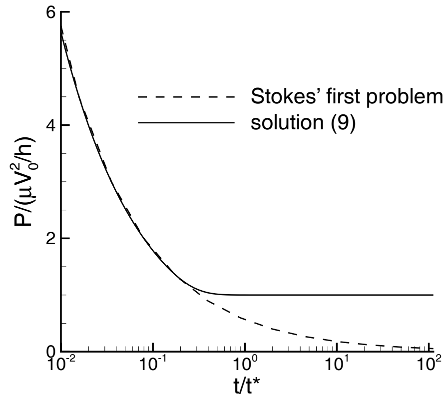

Fig. 3 Time history of the power input from the top wall to water in flows in Fig. 2. $\mu = 1.12 \times 10^{-3}\ \mathrm{N \cdot s/m^2}$.

Remark 2. Computations show that solution (9) converges slowly in terms of n. In Fig.2, $n = 300$ is used. Actually, the slow convergence is true for other solutions hereafter.

The power input from top wall to the fluid is derived as

$$P = \tau(0,t)u(0,t) = \frac{\mu V_0^2}{h}(1 + 2\sum_{n=1}^{\infty} e^{-\nu(\frac{n\pi}{h})^2 t}), \tag{11}$$

where μ is the dynamic viscosity, and $\mu = \rho\nu$, ρ being the density. The variation of power with time is shown in Fig. 3. For comparison, again, the power from top wall to water in the Stokes' first problem is also plotted in the figure. Interestingly, it is seen that in presence of the bottom wall, or a finite depth, the power tends to a constant, and whereas, it goes to zero with time without such wall. Actually, as time goes to infinity, the shear stress at the top wall becomes a constant in the former, and that goes to zero in the latter.

The second simplification happens as $V(t) = V_0 \sin\omega t$, and the solution reads as

$$u(y,t) = (1 + \frac{y}{h})V_0 \sin\omega t + \sum_{n=1}^{\infty} \frac{2V_0}{n\pi} I_n(t) \sin\frac{n\pi y}{h}, \tag{12}$$

where

$$\begin{aligned} I_n(t) &= \frac{\omega^2 \sin\omega t + a\omega\cos\omega t - a\omega e^{-at}}{\omega^2 + a^2}, \\ a &= \nu(\frac{n\pi}{h})^2. \end{aligned} \tag{13}$$

The evolution of solution (12) is plotted in Fig. 4 (a), (b), and (c). It is seen that in the figure the motion of the top plate only causes a thin layer to flow. When time is big enough, the velocity profile is the same as the solution of the classic Stokes' second problem, which can be derived as

$$u(y,t) = V_0 e^{y\sqrt{\omega/2\nu}} \sin(\omega t + \sqrt{\frac{\omega}{2\nu}} y). \tag{14}$$

Actually, in the original Stokes' second problem, $V(t) = V_0 \cos\omega t$, so its solution has a phase difference in comparison with (14) [2]. According to an analysis of the Stokes' second problem, the depth of affected

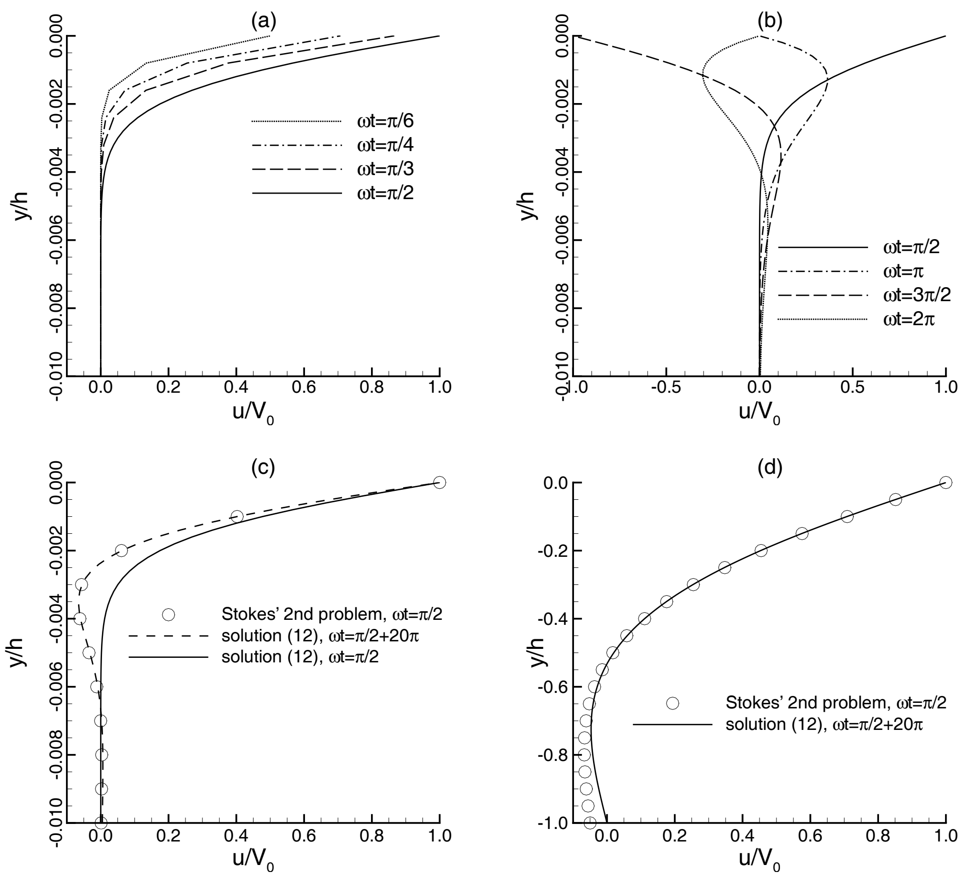

Fig. 4 Water flow driven by an oscillating wall. h=1m, $\omega = 1$ for (a), (b), and (c), and h=0.01m, $\omega = 0.1$ for (d).

layer is proportional to $\sqrt{2\nu/\omega}$, and, when $y = -4\sqrt{\nu/\omega}$, the amplitude of the velocity is only $V_0/20$ [2]. We adopt $h = 0.01\ m$ and $\omega = 0.1$, with $4\sqrt{\nu/\omega} \approx 0.013\ m$, and the effect of the bottom wall becomes more obvious (Fig.4(d)).

The power input from the top wall to the fluid is derived as

$$P = \frac{\mu V_0^2}{h} \sin\omega t(\sin\omega t + 2\sum_{n=1}^{\infty} I_n(t)). \tag{15}$$

The evolution of the power with time is plotted in Fig. 5. The power in the Stokes' second problem can be obtained from Eq. (14), and it is also plotted in the Fig. 5. From this figure, we can see that the power is in a transient process when $t \leq 2\pi$ and it varies periodically as $t > 2\pi$.

Remark 3. As shown in Fig.5, an interesting phenomenon is that the power transferred from the upper wall to the fluid may take negatives values. This happens when the plate moves against the shear stress, or, $(u\partial u/\partial y)_{y=0} < 0$, say, for instance, when $\omega t = 7\pi/8$ in Fig. 6.

Now we take the pressure gradient into account in the governing equation and assume that it is a function of only time t. Let the pressure gradient term be $f(t)$ to account for the pressure term $-(1/\rho)\partial p/\partial x$, ρ being the density. The governing equation in problem (1) is then generalized as

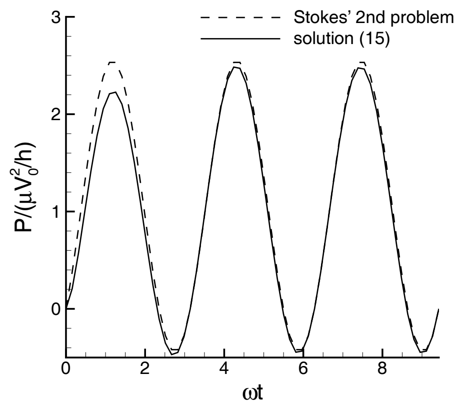

Fig. 5 Time history of the power input from the top wall to the water in flow of Fig. 4(a), (b), and (c).

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} + f(t). \tag{16}$$

The solution of this governing equation associated with the initial-boundary values in Eq. (1) can also be obtained using the method of separating variables, and it reads as

$$u(y,t) = V(t)(1+\frac{y}{h}) + \sum_{n=1}^{\infty} \frac{2}{n\pi}[V(t) - \nu\,(\frac{n\pi}{h})^2 \int_0^t V(\tau)e^{-\nu(\frac{n\pi}{h})^2(t-\tau)}d\tau] \sin\frac{n\pi y}{h} \tag{17}$$
$$+ \sum_{n=1}^{\infty} \{\frac{2}{n\pi}[(-1)^n - 1] \int_0^t f(\tau)e^{-\nu(\frac{n\pi}{h})^2(t-\tau)}d\tau\} \sin\frac{n\pi y}{h}.$$

Obviously, this solution may reduce to a solution for the Poiseuille flow when $V(t) = 0$ and $f(t) = const$. The effect of the pressure term will be discussed in detail in Section 4.

3 Two-layer flows with a step initial velocity distribution

Consider an air-water flow in the horizontal direction between two plates (Fig. 7). The flow is generated by a layer of air that moves with a uniform velocity over a layer of static water. The problem is formulated as

$$\begin{cases} \dfrac{\partial u_a}{\partial t} = \nu_a \dfrac{\partial^2 u_a}{\partial y^2}, \quad 0 < y < H, \\ \dfrac{\partial u_w}{\partial t} = \nu_w \dfrac{\partial^2 u_w}{\partial y^2}, \quad -h < y < 0, \\ u_a(y,0) = V_0, \;\; 0 < y < H; \quad u_w(y,0) = 0, \;\; -h < y < 0, \\ u_a(0,t) = u_w(0,t), \;\; \mu_a \dfrac{\partial u_a}{\partial y}\Big|_{y=0} = \mu_w \dfrac{\partial u_w}{\partial y}\Big|_{y=0}, \\ u_a(H,t) = V_0, \quad u_w(-h,t) = 0, \end{cases} \tag{18}$$

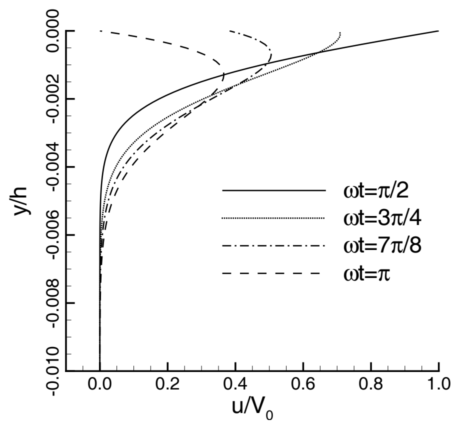

Fig. 6 Evolution of velocity distribution in flow of Fig. 4(a), (b), and (c).

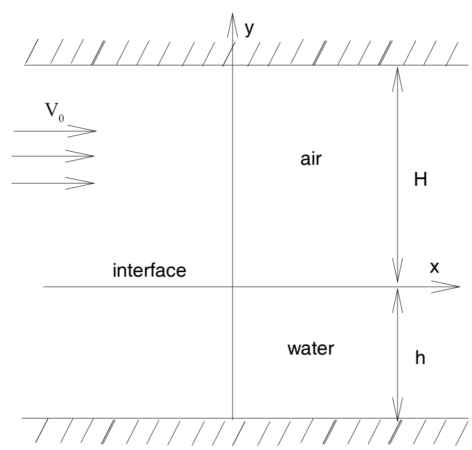

Fig. 7 A schematic representation of a two-layer flow.

where H is the thickness of the air, h is the thickness of the water, V_0 is the initial velocity of the air flow as well as that of the upper wall, and μ is the dynamic viscosity. Subscripts a and w represent the air and the water, respectively. The air-water interface is located at $y = 0$, and initially the velocity of the flow is discontinuous at the interface. The conditions in the fourth line in problem (18) are the connection conditions for the air and the water flow, and they actually require that both velocity and shear stress be continuous across the air-water interface for $t > 0$.

The difficulty in solving the problem is how to implement the connection conditions. In order to overcome the difficulty, we introduce a new function:

$$v(y,t) = au_w(-y,t) + bu_a(\sqrt{\frac{\nu_a}{\nu_w}}y,t), \quad 0 \leq y \leq h, \tag{19}$$

where a, b are two coefficients to be determined. If we assume that

$$H = \sqrt{\frac{\nu_a}{\nu_w}} h, \tag{20}$$

it is readily seen that $v(y,t)$ satisfies

$$\frac{\partial v}{\partial t} = \nu_w \frac{\partial^2 v}{\partial y^2}, \qquad 0 < y < h. \tag{21}$$

Let

$$u^*(y,t) = \begin{cases} v(y,t), & 0 \le y \le h, \\ u_w(y,t), & -h \le y \le 0. \end{cases} \tag{22}$$

Then, u^* satisfies an equation as follows

$$\frac{\partial u^*}{\partial t} = \nu_w \frac{\partial^2 u^*}{\partial y^2}, \qquad y \neq 0. \tag{23}$$

In addition, if we require that $u^*(y,t)$ and its derivative be continuous at $y = 0$, equation (22) yields

$$\begin{aligned} u_w(0,t) &= a u_w(0,t) + b u_a(0,t), \\ \left.\frac{\partial u_w}{\partial y}\right|_{y=0} &= \left.\left(-a\frac{\partial u_w}{\partial y} + b\sqrt{\frac{\nu_a}{\nu_w}}\frac{\partial u_a}{\partial y}\right)\right|_{y=0}. \end{aligned} \tag{24}$$

From Eq. (24) and the connection conditions in Eq. (18), a and b are determined as

$$a = \frac{1-q}{1+q}, \quad b = \frac{2q}{1+q}, \tag{25}$$

where

$$q = \frac{\mu_a}{\mu_w}\sqrt{\frac{\nu_w}{\nu_a}}. \tag{26}$$

On the basis of above transforming, the original problem (18) can be formulated as a new problem that includes a partial differential equation associated with initial and boundary conditions but without any conditions connecting the two fluids. The new problem reads as

$$\begin{cases} \dfrac{\partial u^*}{\partial t} = \nu_w \dfrac{\partial^2 u^*}{\partial y^2}, \\ u^*(y,0) = \dfrac{2qV_0}{1+q}, 0 < y < h;\ u^*(y,0) = 0, -h < y < 0, \\ u^*(-h,t) = 0, \\ u^*(h,t) = \dfrac{2qV_0}{1+q}. \end{cases} \tag{27}$$

Equation (27) is similar to Eq. (1), so it can be also solved using the method of separating variables and the eigenfunction expansion method. The solution to Eq. (27) is derived as

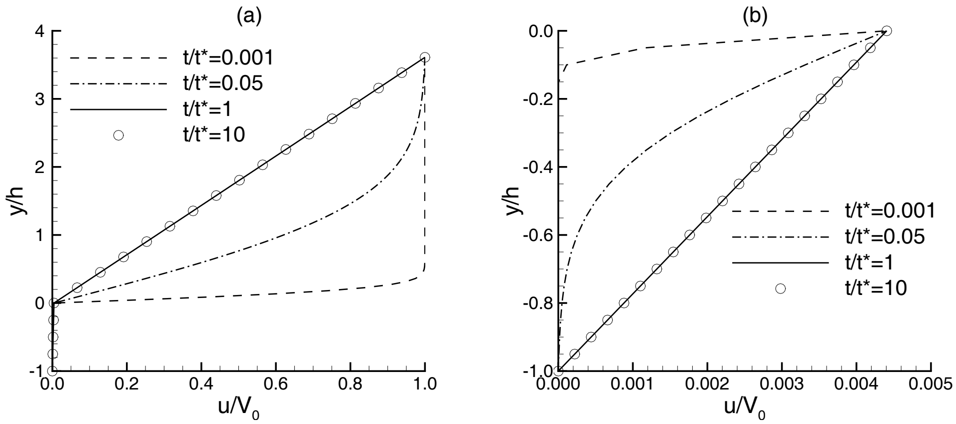

Fig. 8 Air-water flow with an initial step velocity profile. $t^* = h^2/\nu_w$, $\nu_a = 1.46\times10^{-5}$ m^2/s, $\mu_a = 1.79\times10^{-5}$ $\mathrm{N\cdot s/m^2}$, $\nu_w = 1.12\times10^{-6}$ m^2/s, $\mu_w = 1.12\times10^{-3}$ $\mathrm{N\cdot s/m^2}$, $h = 4$ m, and $V_0 = 1$ m/s. (a) Velocity in both air and water. (b) Velocity in water.

$$u^*(y,t) = \frac{qV_0}{1+q}\left(1+\frac{y}{h}+2\sum_{n=1}^{\infty}\frac{1}{n\pi}e^{-\nu_w(\frac{n\pi}{h})^2 t}\sin\frac{n\pi y}{h}\right), \tag{28}$$

or

$$u_a(y,t) = \frac{V_0}{1+q}\left(q+\frac{y}{H}+2\sum_{n=1}^{\infty}\frac{1}{n\pi}e^{-\nu_w(\frac{n\pi}{h})^2 t}\sin\frac{n\pi y}{H}\right), \tag{29}$$

$$u_w(y,t) = \frac{qV_0}{1+q}\left(1+\frac{y}{h}+2\sum_{n=1}^{\infty}\frac{1}{n\pi}e^{-\nu_w(\frac{n\pi}{h})^2 t}\sin\frac{n\pi y}{h}\right). \tag{30}$$

Remark 4. The solution (29) and (30) is not restricted to air and water, and it is applicable to any two-layer of fluids, such as two layers of liquids and two layers of gases.

Remark 5. The velocity distribution has a discontinuity at the interface at $t = 0$, but it becomes continuous when $t > 0$. More interestingly, once $t > 0$, velocity at the air-water interface remains as a constant; letting y=0 in solution (29) and (30), one has

$$u(0,t) = \frac{qV_0}{1+q}. \tag{31}$$

This formula indicates the interface velocity is proportional to the air speed, which is a well-known observation in physical oceanography.

Remark 6. The validity of solution (29) and (30) is restricted by the ratio of air depth to water depth given in Eq. (20). However, it is expected that the behaviors the solution presents general characteristics of the flow.

The solution for velocity profile (29) and (30) at different time is plotted in Fig.8. It is seen that the discontinuity in the initial velocity at the interface disappears once $t > 0$, and, the deformation of the velocity profile propagates towards the upper and lower walls. At about time $t/t^* = 1$, the flow reaches a steady state, which has a linear velocity distribution in both of the air and the water.

Remark 7. It is anticipated that solution (29) and (30) tends to that for a two-layer Couette flow as $t \to 0$, as illustrated in Fig. 8.

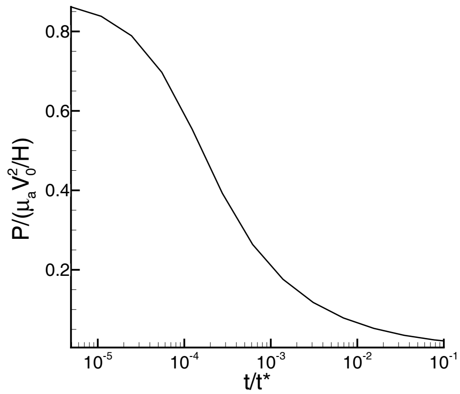

Fig. 9 Time history of the power input from the air to the water in the flow in Fig. 8(a).

The power input from air to water is derived as

$$P = \frac{\mu_w V_0^2 q^2}{h(1+q)^2}(1+2\sum_{n=1}^{\infty} e^{-\nu_w(\frac{n\pi}{h})^2 t}). \tag{32}$$

The variation of the power with time is shown in Fig. 9. It is seen from the figure that the power decreases with time, corresponding to the process that the air reduces its velocity and loses energy to the water and the water gains energy from the air and is set in motion.

4 Effect of pressure gradient

Now let us take pressure into account in problem (18). As before, if we assume that the pressure gradient is a prescribed function of time and be modeled as $f(t)$, the governing equations of the flow in problem (18) are modified as

$$\begin{cases} \dfrac{\partial u_a}{\partial t} = \nu_a \dfrac{\partial^2 u_a}{\partial y^2} + f(t), \quad 0 < y < H, \\ \dfrac{\partial u_w}{\partial t} = \nu_w \dfrac{\partial^2 u_w}{\partial y^2} + f(t), \quad -h < y < 0. \end{cases} \tag{33}$$

We introduce

$$u^*(y,t) = \begin{cases} \dfrac{1-q}{1+q} u_w(-y,t) + \dfrac{2q}{1+q} u_a(\sqrt{\dfrac{\nu_a}{\nu_w}} y, t), \quad 0 \le y \le h, \\ u_w(y,t), \quad -h \le y \le 0. \end{cases} \tag{34}$$

which satisfies

$$\begin{cases} \dfrac{\partial u^*}{\partial t} = \nu_w \dfrac{\partial^2 u^*}{\partial y^2} + f(t), \\ u^*(y,0) = \dfrac{2qV_0}{1+q},\ 0 < y < h; \quad u^*(y,0) = 0,\ -h < y < 0, \\ u^*(-h,t) = 0, \\ u^*(h,t) = \dfrac{2qV_0}{1+q}. \end{cases} \tag{35}$$

We can also solve Eq.(35) by using the method of separating variables and the eigenfunction expansion method, and the solution to Eq.(35) is derived as

$$u^*(y,t) = \frac{qV_0}{1+q}(1 + \frac{y}{h} + 2\sum_{n=1}^{\infty} \frac{1}{n\pi} e^{-\nu_w(\frac{n\pi}{h})^2 t} \sin\frac{n\pi y}{h}) + J_n(y,t), \tag{36}$$

where

$$J_n(y,t) = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{(2n-1)\pi}(\int_0^t f(\tau) e^{-\nu_w(\frac{2n-1}{2h}\pi)^2(t-\tau)} d\tau) \cos\frac{(2n-1)\pi y}{2h}. \tag{37}$$

In particular, the solution is

$$u_a(y,t) = \frac{V_0}{1+q}(q + \frac{y}{H} + 2\sum_{n=1}^{\infty} \frac{1}{n\pi} e^{-\nu_w(\frac{n\pi}{h})^2 t} \sin\frac{n\pi y}{H}) + J_n(y\sqrt{\frac{\nu_w}{\nu_a}}, t), \tag{38}$$

$$u_w(y,t) = \frac{qV_0}{1+q}(1 + \frac{y}{h} + 2\sum_{n=1}^{\infty} \frac{1}{n\pi} e^{-\nu_w(\frac{n\pi}{h})^2 t} \sin\frac{n\pi y}{h}) + J_n(y,t). \tag{39}$$

It is seen that the solution in both of the air and the water consists of two terms; the first term, e.g., solution (29) and (30) describes the motion caused by the wall movement, and the second one, e.g., $J_n(y,t)$, reflects the contribution from the pressure gradient.

Remark 8. Solution (38) and (39) will become a solution for a steady two-layer Poiseuille flow, or, a purely pressure-driven flow, as $V_0 = 0$ and $t \to \infty$.

If we let $f(t) = G = const$, the second item of the solution (38) and (39) can be derived as

$$J_n(y,t) = \frac{16h^2 G}{\nu_w \pi^3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3}(1 - e^{-\nu_w(\frac{2n-1}{2h}\pi)^2 t}) \cos\frac{(2n-1)\pi y}{2H}. \tag{40}$$

Let $C = 16h^2 G/(\nu_w \pi^3)$, solution (38) and (39) for the flow at different time when $C/V_0 = 1$ is shown in Fig.10(a). At a large time, the flow will become steady, and its distributions at different values of C are presented in Fig.10(b).

The power input from the air to the water is derived as

$$P = \frac{\mu_a V_0^2 q}{H(1+q)^2}(1 + 2\sum_{n=1}^{\infty} e^{-\nu_w(\frac{n\pi}{h})^2 t})[1 + \frac{(1+q)C}{qV_0} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3}(1 - e^{-\nu_w(\frac{2n-1}{2h}\pi)^2 t})], \tag{41}$$

which is a modification of power input (32).

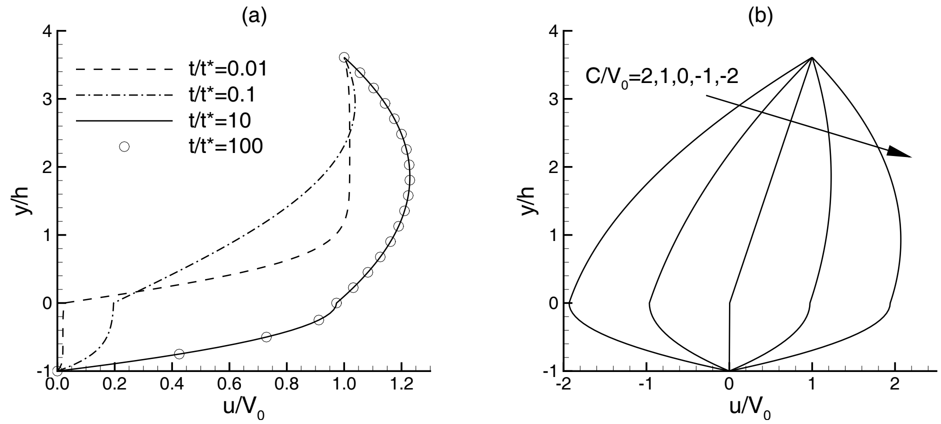

Fig. 10 Air-water flow with a step initial velocity profile and driven by pressure. (a) Variation in time with $C/V_0 = 1$. (b) Steady states with different values of C.

5 Wall-driven two-layer flows

Now we consider an air-water flow driven by motion of the upper wall and pressure gradient:

$$\begin{cases} \dfrac{\partial u_a}{\partial t} = \nu_a \dfrac{\partial^2 u}{\partial y^2} + f(t), \quad 0 < y < H, \\ \dfrac{\partial u_w}{\partial t} = \nu_w \dfrac{\partial^2 u}{\partial y^2} + f(t), \quad -h < y < 0, \\ u_a(y,0) = 0, \quad u_w(y,0) = 0, \\ u_a(0,t) = u_w(0,t), \, \mu_a \dfrac{\partial u_a}{\partial y}\Big|_{y=0} = \mu_w \dfrac{\partial u_w}{\partial y}\Big|_{y=0}, \\ u_a(H,t) = V(t), \quad u_w(-h,t) = 0. \end{cases} \tag{42}$$

The problem can be solved with a procedure similar to those employed in the previous sections. The solution for the air-water flow is obtained as

$$u_a(y,t) = \frac{1}{1+q}\{V(t)(1+q\frac{y}{H}) + 4[\sum_{n=1}^{\infty} K_{2n}(t)(-1)^n \sin\frac{n\pi y}{H} - q\sum_{n=1}^{\infty} K_{2n-1}(t)(-1)^n \cos\frac{(2n-1)\pi y}{2H}]\} \tag{43}$$
$$+ \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{(2n-1)\pi}(\int_0^t f(\tau) e^{-\nu_w(\frac{2n-1}{2h}\pi)^2(t-\tau)} d\tau) \cos\frac{(2n-1)\pi y}{2H},$$

$$u_w(y,t) = \frac{q}{1+q}[V(t)(1+\frac{y}{h}) + 4\sum_{n=1}^{\infty} K_n(t) \sin\frac{n\pi(y+h)}{2h}] \tag{44}$$
$$+ \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{(2n-1)\pi}(\int_0^t f(\tau) e^{-\nu_w(\frac{2n-1}{2h}\pi)^2(t-\tau)} d\tau) \cos\frac{(2n-1)\pi y}{2h},$$

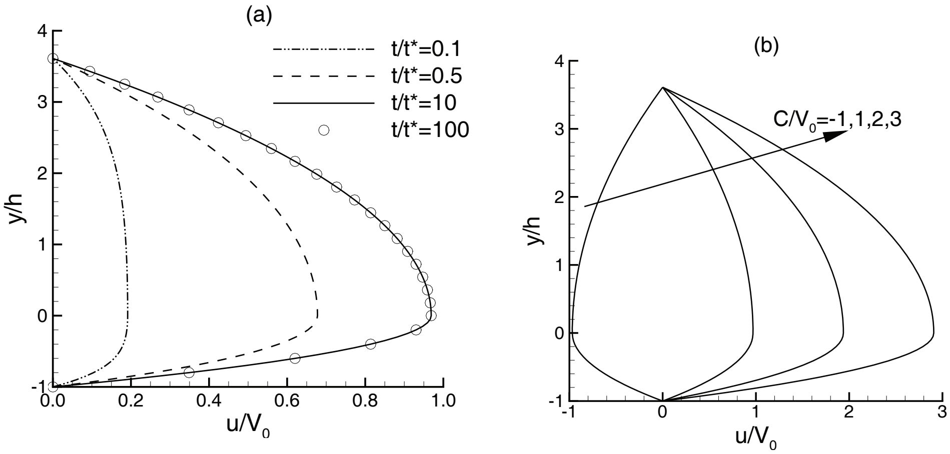

Fig. 11 Pressure driven air-water flow that starts from a static state. (a) Evolution of velocity in time, $C/V_0 = 1$, here $V_0 = \nu_w/h$ and C being defined as before. (b) Velocity profiles at steady states with different pressure gradient.

where

$$K_n(t) = \frac{(-1)^n}{n\pi}[V(t) - \nu_w(\frac{n\pi}{2h})^2 \int_0^t V(\tau)e^{-\nu_w(\frac{n\pi}{2h})^2(t-\tau)}d\tau]. \tag{45}$$

In order to illustrate flow behaviors that solution (43) and (44) describes, a few typical situations of the flow are discussed here. The first air-water flow is started by a pressure gradient. Let $V(t) = 0$, and $f(t) = G$, the solution becomes

$$u_a = \frac{16h^2G}{\nu_w\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^3}\left(1-e^{-\nu_w(\frac{2n-1}{2h}\pi)^2t}\right)\cos\frac{(2n-1)\pi y}{2H}, \tag{46}$$

$$u_w = \frac{16h^2G}{\nu_w\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^3}\left(1-e^{-\nu_w(\frac{2n-1}{2h}\pi)^2t}\right)\cos\frac{(2n-1)\pi y}{2h}. \tag{47}$$

The flow evolves in time and finally reaches a steady state as shown in Fig.11(a). The effect of driven pressure at the steady states can be seen in Fig.11(b). It is interesting to note that the power input from the air to the water is always zero. Although the pressure gradient makes velocity to increase, it does not generate shear stress at the interface; the gradient of the velocity at the interface is always zero so that the shear stress remains as zero. This is

$$\tau(0,t) = \mu_a \left.\frac{\partial u}{\partial y}\right|_{y=0} = \frac{8h^2G}{\nu_w\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{H(2n-1)^2}\left(1-e^{-\nu_w(\frac{2n-1}{2h}\pi)^2t}\right)\sin\frac{(2n-1)\pi\times 0}{2H} = 0. \tag{48}$$

In the second flow, both air and water are initially static and then move due to action of the top wall. Let the top wall velocity be $V(t) = V_0(1-e^{-t})$, but ignore the effect of pressure or set $f(t) = 0$. The solution becomes

$$u_a(y,t) = \frac{1}{1+q}\{V_0(1-e^{-t})(1+q\frac{y}{H}) + 4[\sum_{n=1}^{\infty}K_{2n}(t)(-1)^n\sin\frac{n\pi y}{H} \tag{49}$$
$$-q\sum_{n=1}^{\infty}K_{2n-1}(t)(-1)^n\cos\frac{(2n-1)\pi y}{2H}]\},$$

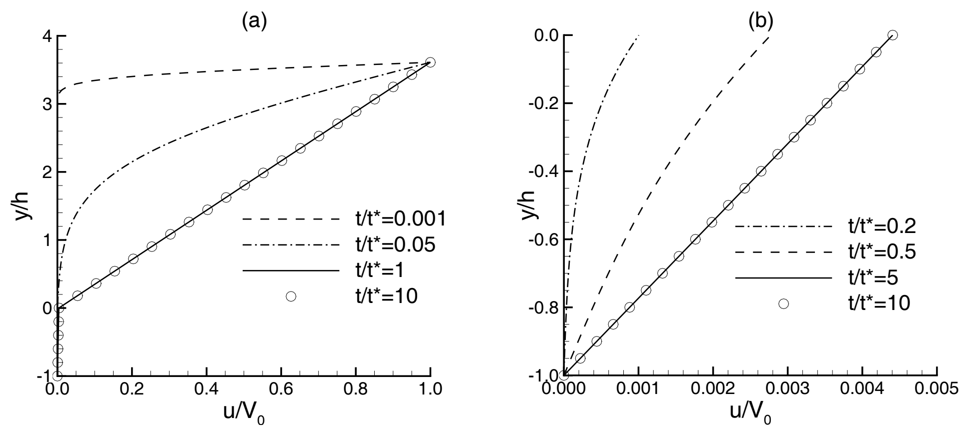

Fig. 12 Air-water flow dragged by the top wall with gradual motion. (a) Velocity in both air and water. (b) Velocity in water.

$$u_w(y,t)=\frac{q}{1+q}[V_0(1-e^{-t})(1+\frac{y}{h})+4\sum_{n=1}^{\infty}K_n(t)\sin\frac{n\pi(y+h)}{2h}], \tag{50}$$

where

$$K_n(t)=\frac{V_0(-1)^n}{n\pi}[\frac{1}{1-\nu_w(n\pi/(2h))^2}(e^{-\nu_w(\frac{n\pi}{2h})^2t}-e^{-t})]. \tag{51}$$

Solution (49) and (50) is always continuous (Fig.12). When time tends to infinity, it is readily seen that in solution (49) and (50) the velocity at the interface reduces to Eq. (31). The power input from the air to the water is

$$P=\frac{\mu_w}{h}\frac{q^2}{(1+q)^2}[V_0(1-e^{-t})+4\sum_{n=1}^{\infty}K_n(t)\sin\frac{n\pi}{2}][V_0(1-e^{-t})+2\sum_{n=1}^{\infty}n\pi K_n(t)\cos\frac{n\pi}{2}]. \tag{52}$$

The variation of power input from the air to the water is shown in Fig.13. At the beginning, the velocity and its gradient at the interface are zero and thus the power is zero also. Then, gradually the fluids are set in motion at the interface, and the power increases from zero to a constant.

In the third flow, the both fluids are initially static but the top wall velocity is oscillating at speed $V(t)=V_0\sin\omega t$. Now the solution becomes

$$\begin{aligned} u_l(y,t)=\frac{1}{1+q}\{&V_0\sin\omega t(1+q\frac{y}{H})+4[\sum_{n=1}^{\infty}K_{2n}(t)(-1)^n\sin\frac{n\pi y}{H} \\ &-q\sum_{n=1}^{\infty}K_{2n-1}(-1)^n\cos\frac{(2n-1)\pi y}{2H}]\}, \end{aligned} \tag{53}$$

$$u_g(y,t)=\frac{q}{1+q}[V_0\sin\omega t(1+\frac{y}{h})+4\sum_{n=1}^{\infty}K_n(t)\sin\frac{n\pi(y+h)}{2h}], \tag{54}$$

where

$$K_n(t)=\frac{(-1)^nV_0}{n\pi}[\sin\omega t-\nu_g(\frac{n\pi}{2h})^2\frac{\omega e^{-\nu_g(\frac{n\pi}{2h})^2t}+\nu_g(\frac{n\pi}{2h})^2\sin\omega t-\omega\cos\omega t}{\omega^2+\nu_g^2(n\pi/(2h))^4}]. \tag{55}$$

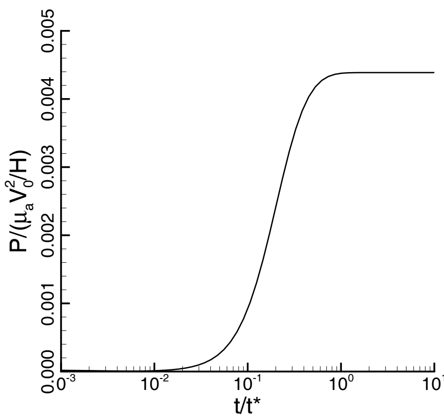

Fig. 13 Variation of power input from air to water in the flow in Fig. 12(a).

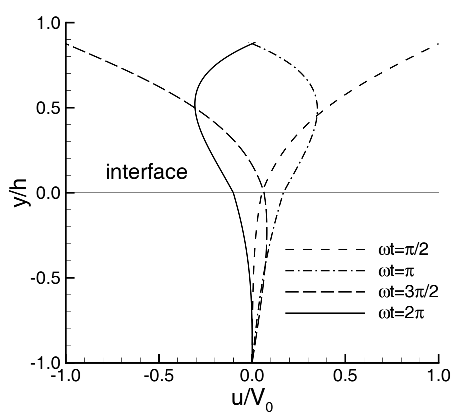

Fig. 14 Solution for Lube-Glycerin flow driven by an oscillating wall. $\mu_l = 0.999\ \mathrm{N\cdot s/m^2}$, $\nu_l = 0.00112\ \mathrm{m^2/s}$, $\mu_g = 1.85\ \mathrm{N\cdot s/m^2}$, and $\nu_g = 0.00146\ \mathrm{m^2/s}$.

As shown in Fig.4, the top plate can only affect very thin layer of the upper fluid and the depth of affected layer is related to the viscosities of the fluids. In order to achieve stronger motion in the both fluids, lube and glycerin are selected as the upper and lower layer fluid, respectively. Let the depth of glycerin H to be 10cm, and

$$q = \frac{\mu_l}{\mu_g}\sqrt{\frac{\nu_g}{\nu_l}}. \tag{56}$$

where subscript l and g represent lube and glycerin, respectively. The transient process is shown in Fig.14. The power input from lube to glycerin is

$$P = \frac{\mu_g}{h}\frac{q^2}{(1+q)^2}(V_0\sin\omega t + 4\sum_{n=1}^{\infty}K_n(t)\sin\frac{n\pi}{2})(V_0\sin\omega t + 2\pi\sum_{n=1}^{\infty}nK_n(t)\cos\frac{n\pi}{2}). \tag{57}$$

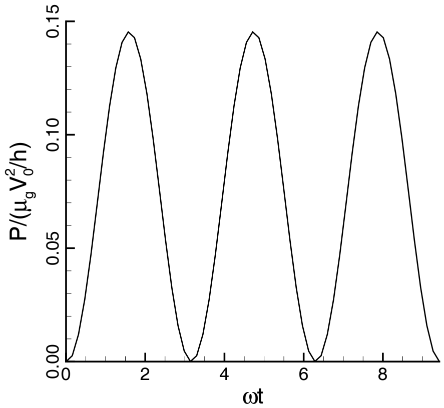

Fig. 15 Variation of power input from the lube to the glycerin in the flow of Fig. 14.

The variation of power input is shown in Fig.15.

6 Conclusions

In this paper, a group of exact solutions are presented for extended Stokes' problems. The problems are solved using the method of separating variables and the eigenfunction expansion method, and the solutions are obtained in form of Fourier series. It is shown that the derived solutions may degenerate to solutions for the classic Stokes' problems, the Couette flow, and the Poiseuille flow under corresponding initial and boundary conditions. Although the extended problems are linear, the solutions show that the flows involve complex as well as interesting physical processes with regard to velocity profiles and power transferring at fluid interfaces. We anticipate that the solutions and analyses in this article will shed light in understanding of actual problems such as free surface flows in oceans.

Acknowledgments

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References

[1] Stokes, G.G. (1851), On the effect of the internal friction of fluids on the motion of pendulums, *Transaction of the Cambridge Philosophical Society*, **9**, 8–106.
[2] Kundu, P. K. (1990), *Fluid Mechanics*, Academic Press, New York.
[3] Schlichting, H. (1968), *Boundary Layer Theory,*, Sixth edition, McGraw-Hill, New York.
[4] Raffaele, F and Carl, W. (2009), Ocean circulation kinetic energy: reservoirs, sources, and sinks, *Annual Review of Fluid Mechanics*, **41**,253–282.
[5] Ozinsky, A.E. and Ekama, G.A. (1995), Secondary setting-tank modeling and design. Part 1: Review of theoretical and practical developments, *Water SA*, **21**, 325–332.
[6] Iorio, C.S., Goncharova, O. and Kabov, O. (2011), Influence of boundaries on shear-driven flow of liquids in open cavities, *Microgravity Science and Technology*, **23**, 373–379.

[7] Erdogan, M.E. (2000), A note on an unsteady flow of a viscous fluid due to an oscillating plane wall, *International Journal of Non-Linear Mechanics* **35**, 1–6.
[8] Liu, C.M. (2008), Complete solutions to extended Stokes' problems, *Mathematical Problems in Engineering*, **2008**, 754262, doi:10.1155/2008/754262.
[9] Tang, H.S., Zhang, L.Z., Maa, J.P.-Y., Li, H., Jiang, C.B. and Hussain, R. (2013), Fluid driven by tangential velocity and shear stress: Mathematical analysis, numerical experiment, and implication to surface flow. *Mathematical Problems in Engineering*, In press.
[10] Muzychka, Y.S. and Yovanovich, M.M. (2010), Unsteady viscous flows and Stokes's first problem, *International Journal of Thermal Sciences*, **49**, 820–828.
[11] Tan, W. C. and Masuoka, T. (2005), Stokes first problem for an Oldroyd-B fluid in a porous half space, *Physics of Fluids*, **17**, 023101.
[12] Ali, F., Norzieha, M., Sharidan, S., Khan, I. and Hayat, T. (2012), New exacts olutions of Stokes' second problem for an MHD second grade fluid inaporousspace, *International Journal of Non-Linear Mechanics*, **47**, 521–525.
[13] Evans, L.C. (1998), *Partial Differential Equations*, American Mathematical Society, Provindence.